POISSON BRACKET

⇒ Poisson Bracket } Two brackets P.B — [ ]
Langrange's Bracket } in syllabus. L.B — { }

⇒ **POISSON BRACKET** :— Let $G(q,p)$ be a dynamical quantity with (q,p) as canonical co-ordinates. The time derivative of G is given by

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \sum_{k=1}^{n}\left[\frac{\partial G}{\partial q_k}\dot{q}_k + \frac{\partial G}{\partial p_k}\dot{p}_k\right]$$

Since (q,p) are canonical co-ordinates

$$\therefore \quad \dot{q}_k = \frac{\partial H}{\partial p_k} \quad ; \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \underbrace{\sum_{k=1}^{n}\left[\frac{\partial G}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial G}{\partial p_k}\frac{\partial H}{\partial q_k}\right]}$$

$$= \frac{\partial G}{\partial t} + [G,H]_{P.B}$$

where $[G,H]_{P.B} = \sum_{k=1}^{n}\left[\frac{\partial G}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial G}{\partial p_k}\frac{\partial H}{\partial q_k}\right]$

If G is not an explicit function of t, then

$$\frac{\partial G}{\partial t} = 0$$

and we have $\frac{dG}{dt} = [G,H]$

Again if G is constant of motion, then

$\frac{dG}{dt} = 0$ and we have $[G, H] = 0$

In general, if A & B are two dynamical quantities (function of q and p or may be function of time), then

$$[A, B] = \sum_{k=1}^{n} \left[\frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial q_k} \right]$$

We have Certain Properties :-

i) $[A + B, C] = [A, C] + [B, C]$

ii) $[A, B] = -[B, A]$

iii) $[AB, C] = A[B, C] + [A, C]B$ → Prove it.

Proof (iii)

$$[AB, C] = \sum_{k=1}^{n} \left[\frac{\partial (AB)}{\partial q_k} \frac{\partial (C)}{\partial p_k} - \frac{\partial (AB)}{\partial p_k} \frac{\partial C}{\partial q_k} \right]$$

$$= \sum_{k=1}^{n} \left[\left(A \frac{\partial B}{\partial q_k} + B \frac{\partial A}{\partial q_k} \right) \frac{\partial C}{\partial p_k} - \left(A \frac{\partial B}{\partial p_k} + \frac{\partial A}{\partial p_k} \cdot B \right) \frac{\partial C}{\partial q_k} \right]$$

$$= \sum_{k=1}^{n} \left[A \left(\frac{\partial B}{\partial q_k} \frac{\partial C}{\partial p_k} - \frac{\partial B}{\partial p_k} \frac{\partial C}{\partial q_k} \right) + B \left(\frac{\partial A}{\partial q_k} \frac{\partial C}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial C}{\partial q_k} \right) \right]$$

$$= A[B, C] + [A, C]B$$

Proved

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(iv) $[ABC, D] = [XC, D]$ where $X = AB$

$= X[C,D] + [X,D]C$

$= AB[C,D] + [AB,D]C$

$= AB[C,D] + (A[B,D] + [A,D]B)C$

$= AB[C,D] + A[B,D]C + [A,D]BC$

⇒ Fundamental Poisson Brackets :-

① $[q_i, q_j] = [p_i, p_j] = 0$

② $[q_i, p_j] = \delta_{ij}$

Proof ① $[q_i, q_j] = \sum_{k=1}^{n}\left[\frac{\partial q_i}{\partial q_k}\frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial q_j}{\partial q_k}\right]$

$= \sum_{k=1}^{n}\left[\frac{\partial q_i}{\partial q_k}\cdot 0 - \frac{\partial q_i}{\partial p_k}\frac{\partial q_j}{\partial q_k}\right]$

$= \sum_{k=1}^{n}\left[\left(\frac{\partial q_i}{\partial q_k}\cdot 0\right) - \left(0\cdot\frac{\partial q_j}{\partial q_k}\right)\right]$

$= 0$ — $\left[\because q \,\&\, p \text{ are independent so } \frac{\partial q_i}{\partial p_k} = 0 \,\&\, \frac{\partial q_j}{\partial p_k} = 0\right]$

Similarly $[p_i, p_j] = 0$

② $[q_i, p_j] = \sum_{k=1}^{n}\left[\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k}\right]$

$= \sum_{k=1}^{n}\left[\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - 0\cdot 0\right]$ — $[q \,\&\, p \text{ are independent}]$

$= \sum \delta_{ik}\,\delta_{jk}$

$= \delta_{ij}$

where δ_{ij} is Kronecker delta and is defined as

$$\delta_{ij} = \begin{cases} 0 , & i \neq j \\ 1 , & i = j \end{cases}$$

Remark

$$[q_i, H] = \sum \left[\frac{\partial q_i}{\partial q_k} \frac{\partial H}{\partial p_k} - \cancelto{0}{\frac{\partial q_i}{\partial p_k}} \frac{\partial H}{\partial q_k} \right]$$

$$= \sum \delta_{ik} \frac{\partial H}{\partial p_k}$$

$$= \frac{\partial H}{\partial p_i} = \dot{q}_i$$

$$[p_i, H] = \sum \left[\cancelto{0}{\frac{\partial p_i}{\partial q_k}} \frac{\partial H}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial H}{\partial q_k} \right]$$

$$= \sum \left(- \frac{\partial p_i}{\partial p_k} \frac{\partial H}{\partial q_k} \right)$$

$$= \sum \left(- \delta_{ik} \frac{\partial H}{\partial q_k} \right) = - \frac{\partial H}{\partial q_i} = \dot{p}_i$$

Thus the Hamilton's canonical equations in terms of P.B. are

$$[q_i, H] = \dot{q}_i \quad , \quad [p_i, H] = \dot{p}_i$$

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NOTE

1. $[u, u] = 0$

2. $[u, v] = -[v, u]$ (antisymmetry)

3. $[au + bv, w] = a[u, w] + b[v, w]$ (linearity)
   where a & b are constants.

4. If c is constant, then
   $[c, x] = 0$

5. $c[A, B] = [cA, B]$

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Jacobi Identity

→ If X, Y, Z are dynamical quantities; (p, q) are canonical co-ordinates, then

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Proof. We shall prove that

$$[X, [Y, Z]] + [Y, [Z, X]] = -[Z, [X, Y]]$$

$$\text{L.H.S.} = [X, [Y, Z]] + [Y, [Z, X]]$$

$$= [X, [Y, Z]] - [Y, [X, Z]]$$

$$= \sum_k \left(\left[X, \frac{\partial Y}{\partial q_k} \frac{\partial Z}{\partial p_k}\right] - \left[X, \frac{\partial Y}{\partial p_k} \frac{\partial Z}{\partial q_k}\right] \right.$$

As
$[X, AB] = A[X, B] + [X, A]B$

$$\left. - \left[Y, \frac{\partial X}{\partial q_k} \frac{\partial Z}{\partial p_k}\right] + \left[Y, \frac{\partial X}{\partial p_k} \frac{\partial Z}{\partial q_k}\right] \right)$$

$$= \sum_k \left(\frac{\partial Y}{\partial q_k}\left[X, \frac{\partial Z}{\partial p_k}\right] + \left[X, \frac{\partial Y}{\partial q_k}\right] \frac{\partial Z}{\partial p_k} \right.$$

$$- \frac{\partial Y}{\partial p_k}\left[X, \frac{\partial Z}{\partial q_k}\right] - \left[X, \frac{\partial Y}{\partial p_k}\right] \frac{\partial Z}{\partial q_k}$$

$$- \frac{\partial X}{\partial q_k}\left[Y, \frac{\partial Z}{\partial p_k}\right] - \left[Y, \frac{\partial X}{\partial q_k}\right] \frac{\partial Z}{\partial p_k}$$

$$\left. + \frac{\partial X}{\partial p_k}\left[Y, \frac{\partial Z}{\partial q_k}\right] + \left[Y, \frac{\partial X}{\partial p_k}\right] \frac{\partial Z}{\partial q_k} \right)$$

Consider the terms

$$\sum \left(\left[X, \frac{\partial Y}{\partial q_k}\right] \frac{\partial Z}{\partial p_k} - \left[X, \frac{\partial Y}{\partial p_k}\right] \frac{\partial Z}{\partial q_k} - \left[Y, \frac{\partial X}{\partial q_k}\right] \frac{\partial Z}{\partial p_k} + \left[Y, \frac{\partial X}{\partial p_k}\right] \frac{\partial Z}{\partial q_k} \right)$$

$$= \sum \left(\left\{ \left[X, \frac{\partial Y}{\partial q_k}\right] + \left[\frac{\partial X}{\partial q_k}, Y\right] \right\} \frac{\partial Z}{\partial p_k} - \left\{ \left[X, \frac{\partial Y}{\partial p_k}\right] + \left[\frac{\partial X}{\partial p_k}, Y\right] \right\} \frac{\partial Z}{\partial q_k} \right)$$

$$= \sum \left(\frac{\partial}{\partial q_k}[X, Y] \frac{\partial Z}{\partial p_k} - \frac{\partial}{\partial p_k}[X, Y] \frac{\partial Z}{\partial q_k} \right)$$

$$= \left[[X, Y], Z\right] = -\left[Z, [X, Y]\right]$$

The remaining terms are

$$R = \sum \left(\frac{\partial Y}{\partial q_k} \left[X, \frac{\partial Z}{\partial p_k}\right] - \frac{\partial Y}{\partial p_k} \left[X, \frac{\partial Z}{\partial q_k}\right] - \frac{\partial X}{\partial q_k} \left[Y, \frac{\partial Z}{\partial p_k}\right] + \frac{\partial X}{\partial p_k} \left[Y, \frac{\partial Z}{\partial q_k}\right] \right)$$

<u>To show : $R = 0$</u>

$$R = \sum_k \sum_r \left\{ \frac{\partial Y}{\partial q_k} \cdot \left(\frac{\partial X}{\partial q_r} \frac{\partial^2 Z}{\partial p_r \partial p_k} - \frac{\partial X}{\partial p_r} \frac{\partial^2 Z}{\partial q_r \partial p_k} \right) \right.$$

$$- \frac{\partial Y}{\partial p_k} \left(\frac{\partial X}{\partial q_r} \frac{\partial^2 Z}{\partial p_r \partial q_k} - \frac{\partial X}{\partial p_r} \frac{\partial^2 Z}{\partial q_r \partial q_k} \right)$$

$$- \frac{\partial X}{\partial q_k} \left(\frac{\partial Y}{\partial q_r} \frac{\partial^2 Z}{\partial p_r \partial p_k} - \frac{\partial Y}{\partial p_r} \frac{\partial^2 Z}{\partial q_r \partial p_k} \right)$$

$$\left. + \frac{\partial X}{\partial p_k} \left(\frac{\partial Y}{\partial q_r} \frac{\partial^2 Z}{\partial q_k \partial p_r} - \frac{\partial Y}{\partial p_r} \frac{\partial^2 Z}{\partial q_r \partial q_k} \right) \right\}$$

$$= \sum_{k} \sum_{r} \Big\{ \frac{\partial^2 Z}{\partial p_k \partial p_r} \left(\frac{\partial Y}{\partial q_k} \cdot \frac{\partial X}{\partial q_r} - \frac{\partial Y}{\partial q_r} \frac{\partial X}{\partial q_k} \right)$$

$$+ \frac{\partial^2 Z}{\partial q_k \partial p_r} \left(-\frac{\partial Y}{\partial p_k} \frac{\partial X}{\partial q_r} + \frac{\partial X}{\partial p_k} \frac{\partial Y}{\partial q_r} \right)$$

$$+ \frac{\partial^2 Z}{\partial q_k \partial q_r} \left(\frac{\partial Y}{\partial p_k} \frac{\partial X}{\partial p_r} - \frac{\partial X}{\partial p_k} \frac{\partial Y}{\partial p_r} \right)$$

$$+ \frac{\partial^2 Z}{\partial p_k \partial q_r} \left(-\frac{\partial Y}{\partial q_k} \frac{\partial X}{\partial p_r} + \frac{\partial X}{\partial q_k} \frac{\partial Y}{\partial p_r} \right) \Big\}$$

$= 0$ (Differentiation can be interchanged.)

Hence, $[X, [Y, Z]] + [Y, [Z, X]] = -[Z, [X, Y]]$

ie $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Proved

POISSON THEOREM

→ If two dynamical quantities are constants of motion then their P.B. is also a constant of motion.

Proof (Step 1) Let f and g be two dynamical quantities such that

$$\frac{df}{dt} = 0 = \frac{dg}{dt} \quad \text{(constant of motion)}$$

To show that $\frac{d}{dt}[f, g] = 0$ (Constant of Motion)

Step-2 In general we know that

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_k \left[\frac{\partial f}{\partial q_k}\dot{q}_k + \frac{\partial f}{\partial p_k}\dot{p}_k\right]$$

But $\dot{q}_k = \frac{\partial H}{\partial p_k}$, $\dot{p}_k = -\frac{\partial H}{\partial q_k}$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H]$$

Similarly, $\frac{dg}{dt} = \frac{\partial g}{\partial t} + [g, H]$

Step-3 Now $\frac{d}{dt}[f,g] = \frac{\partial}{\partial t}[f,g] + [[f,g],H]$

$$= \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right] - [H,[f,g]]$$

$$= \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right] + [f,[g,H]] + [g,[H,f]]$$

$\because$ $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

Put $Z = H$, $X = f$, $Y = g$

$\Rightarrow [f,[g,H]] + [g,[H,f]] + [H,[f,g]] = 0$

$\Rightarrow [f,[g,H]]$

$$= \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right] + [f,[g,H]] + [[f,H],g]$$

$$= \left[\frac{\partial f}{\partial t} + [f,H], g\right] + \left[f, \frac{\partial g}{\partial t} + [g,H]\right]$$

$$= \left[\frac{df}{dt}, g\right] + \left[f, \frac{dg}{dt}\right]$$

$$= [0, g] + [f, 0]$$

$$= 0$$

H.P.

Remark q, p, H are three dynamical quantities. Applying Jacobi's identity, we have

$$[q, [p, H]] + [p, [H, q]] + [H, [q, p]] = 0$$

Now, $[q, p] = 1$, $[p, H] = \dot{p}$, $[q, H] = \dot{q}$

$$\Rightarrow [q, \dot{p}] + [p, -\dot{q}] + [H, 1] = 0$$

$$\Rightarrow [q, \dot{p}] + [\dot{q}, p] = [1, H]$$

Now $$[1, H] = \sum_k \left[\frac{\partial(1)}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial(1)}{\partial p_k}\frac{\partial H}{\partial q_k}\right] = 0$$

$$\Rightarrow [q, \dot{p}] + [\dot{q}, p] = 0$$

$$\Rightarrow \frac{d}{dt}[q, p] = 0$$

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